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Semigroups of operators with nonnegative diagonals[☆]

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ABSTRACT

If $\mathcal{S}$ is an irreducible semigroup of complex matrices and if every member of $\mathcal{S}$ has nonnegative diagonal entries, then is $\mathcal{S}$ simultaneously similar to a semigroup of nonnegative matrices? Partial affirmative answers to this and its infinite-dimensional analogue are given and several counterexamples presented.

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1. Introduction

Multiplicative semigroups of nonnegative matrices and, more generally, operators on function spaces that preserve nonnegativity have been known to have interesting properties. The well-known Perron–Frobenius theorem and its extensions, for example, show how spectral and fixed-point results can be obtained from nonnegativity. (For a treatment of some of these results see [2].)

This work was motivated by the search for sufficient conditions for simultaneous similarity of a given semigroup in $M_n(\mathbb{C})$ or in $M_n(\mathbb{R})$ to a semigroup of nonnegative matrices, i.e., one in $M_n(\mathbb{R}^+)$. It was shown in [1] that if $\mathcal{G}$ is an irreducible group in $M_n(\mathbb{C})$, i.e., a group with no common invariant subspace other than the obvious ones $\{0\}$ and $\mathbb{C}^n$, with the property that the diagonal of every member consists of nonnegative numbers, then $\mathcal{G}$ is in fact (simultaneously) similar to a group in $M_n(\mathbb{R}^+)$ (and thus a group of weighted permutations, by known results). Extending this result from groups to semigroups soon proved to be impossible (see Theorems 1 and 2). Our joint efforts, however, resulted in certain instructive and intricate counterexamples and some partial affirmative results of interest. We present them in this note.

2. Operators on $\mathbb{R}^n$ with $n \geqslant 3$

In this section we introduce two negative results which demonstrate how difficult (and how different from the group case) the semigroup case of the problem is. The only affirmative result known so far is that an irreducible semigroup in $M_n(\mathbb{R})$ with nonnegative diagonals in which every nonzero member has rank one is similar to a semigroup in $M_n(\mathbb{R}^+)$ (see [1, Theorem 3.1]).

Theorem 1. *There exists an irreducible semigroup in $M_3(\mathbb{R})$ that is not similar to a semigroup of nonnegative matrices, although it consists of nonnegative matrices with the exception of one matrix which has only one negative entry (in position $(1,3)$).*

Proof. Let X be the maximal "ice cream" cone around the vector $e = (1,1,1)^T$ that is contained in the standard nonnegative cone $\mathbb{R}^3_+$, i.e.,

$$X = \{x \in \mathbb{R}^3_+ : e^T x \geqslant \sqrt{2}\,\|x\|\},$$

where $\|\cdot\|$ denotes the Euclidean vector norm. It is easy to see that its dual cone is

$$X^d = \{y \in \mathbb{R}^3 : x^T y \geqslant 0 \text{ for all } x \in X\} = \{y \in \mathbb{R}^3_+ : e^T y \geqslant \|y\|\}.$$

Let $\mathcal{S}_1$ be the irreducible semigroup of all rank-one nonnegative matrices xy^T with $x, y \in X$. Define an orthonormal basis of $\mathbb{R}^3$ by

$$e_1 = \frac{1}{\sqrt{3}}(1,1,1)^T, \quad e_2 = \frac{1}{\sqrt{2}}(1,-1,0)^T, \quad e_3 = \frac{1}{\sqrt{6}}(1,1,-2)^T.$$

Let U be the change of basis matrix from this basis to the standard one, i.e.,

$$U = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \end{pmatrix}.$$

Define

$$S_0 = U\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}U^T = \frac{1}{6}\begin{pmatrix} 2+\sqrt{3} & 2+\sqrt{3} & 2-2\sqrt{3} \\ 2-\sqrt{3} & 2-\sqrt{3} & 2+2\sqrt{3} \\ 2 & 2 & 2 \end{pmatrix}.$$

Then the set $\mathcal{S} = \mathcal{S}_1 \cup \{S_0\}$ is a semigroup, since S_0 and S_0^T leave the cone X invariant and since $S_0^2 = e_1 e_1^T \in \mathcal{S}_1$. Clearly, $\mathcal{S}$ consists of nonnegative matrices with the exception of the matrix S_0 which has only one negative entry in position $(1,3)$.

It remains to show that $\mathcal{S}$ is not similar to a semigroup of nonnegative matrices. Assume the contrary. Then there exists an invertible matrix $T \in M_3(\mathbb{R})$ such that the cone $K = T(\mathbb{R}^3_+)$ is invariant under all members of $\mathcal{S}$. Since the rank-one matrices of $T^{-1}\mathcal{S}T$ are nonnegative, we must have either $T^{-1}(X) \subseteq \mathbb{R}^3_+$ or $T^{-1}(X) \subseteq -\mathbb{R}^3_+$. Replacing T by $-T$ (if necessary) we may assume that $T^{-1}(X) \subseteq \mathbb{R}^3_+$, so that $X \subseteq K$. Choose $x \in K$. If $y \in X$, then $(y^Tx)y = (yy^T)x \in K$, as K is invariant under $yy^T \in \mathcal{S}$. Since $y \in K$, we must have that $y^Tx \geqslant 0$. It follows that $x \in X^d$, so that $K \subseteq X^d$. Thus, we have $X \subseteq K \subseteq X^d$.

Now, we claim that the extreme rays of K are mutually orthogonal. To show this, let Σ be the plane through $(1, 1, 1)$ that is orthogonal to the vector e_1. Then $X \cap \Sigma$ and $X^d \cap \Sigma$ are concentric circles of radii $\frac{\sqrt{3}}{\sqrt{2}}$ and $\sqrt{6}$, respectively. Observe that the ratio of these radii is equal to 2. Since $K \cap \Sigma$ is a triangle lying in between these circles, we conclude that $K \cap \Sigma$ is an equilateral triangle, and this implies our claim.

It follows that T is of the form QPD, where D is a positive diagonal matrix, P a permutation matrix and Q a rotation matrix around the vector e_1. Since P, D, P^{-1} and D^{-1} leave the standard cone $\mathbb{R}^3_+$ invariant, there is no loss of generality in assuming that $T = Q$, that is,

$$T = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} U^T$$

for some $\varphi \in [0, 2\pi)$.

However, computing the entries in positions $(1, 3)$, $(2, 1)$ and $(3, 2)$ of the matrix $A = T^{-1}S_0T$ we obtain

$$a_{13} = \frac{1}{6}\left(2 - \sqrt{3} - 2\sin\left(2\varphi + \frac{\pi}{3}\right)\right),$$
$$a_{21} = \frac{1}{6}\left(2 - \sqrt{3} + 2\sin(2\varphi)\right),$$
$$a_{32} = \frac{1}{6}\left(2 - \sqrt{3} - 2\sin\left(2\varphi - \frac{\pi}{3}\right)\right).$$

The matrix A cannot be nonnegative, since a_{13} is negative for $\varphi \in \left[-\frac{\pi}{12} + k\pi, \frac{3\pi}{12} + k\pi\right]$, $k \in \mathbb{Z}$, a_{32} is negative for $\varphi \in \left[\frac{3\pi}{12} + k\pi, \frac{7\pi}{12} + k\pi\right]$, $k \in \mathbb{Z}$, and a_{21} is negative for $\varphi \in \left[\frac{7\pi}{12} + k\pi, \frac{11\pi}{12} + k\pi\right]$, $k \in \mathbb{Z}$. This contradiction completes the proof. $\square$

For the next result we recall the definition of a circulant matrix. Let C be the permutation matrix of size n corresponding to the largest cycle:

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

A circulant matrix is a matrix of the form $p(C)$, where p is a polynomial with complex coefficients. The set of all circulant matrices is a commutative algebra that is closed under transposition, since $C^T = C^{n-1}$.

Theorem 2. *For each integer $n \geqslant 3$ there exists an irreducible semigroup $\mathcal{S}$ in $M_n(\mathbb{R})$ with the properties*:

(1) *The diagonal entries of each member are nonnegative*;
(2) *Every member of rank more than one is a circulant matrix*;

(3) *$\mathcal{S}$ is symmetric, i.e., $S^T \in \mathcal{S}$ whenever $S \in \mathcal{S}$;*
(4) *$\mathcal{S}$ is not similar to a semigroup of nonnegative matrices.*

Proof. Let X be the ice cream cone around the vector $e = (1, 1, \ldots, 1)^T$:

$$X = \{x \in \mathbb{R}^n_+ : e^T x \geqslant c\,\|x\|\},$$

where $c \in \left[\sqrt{n-1}, \sqrt{n}\right)$, so that X is contained in the standard nonnegative cone $\mathbb{R}^n_+$. Let $\mathcal{S}_1$ be the irreducible semigroup of all rank-one nonnegative matrices xy^T with $x, y \in X$. We will enlarge $\mathcal{S}_1$ to a semigroup $\mathcal{S}$ having the desired properties. If we define $P = \frac{1}{n}ee^T$, then $P^2 = P = PC = CP$ and $nP = I + C + C^2 + \cdots + C^{n-1}$. Define a circulant matrix $A = 2P - C$. Then $A^TA = AA^T = I$, so that A is an isometry. For each $x \in \mathbb{C}^n$ we have $e^T(Ax) = (A^Te)^Tx = e^Tx$ and $e^T(A^Tx) = (Ae)^Tx = e^Tx$, from which it follows that both A and A^T leave X invariant. Since $A^T = A^{-1}$ is also a circulant matrix, the set

$$\mathcal{S} = \mathcal{S}_1 \cup \{A^k : k \in \mathbb{Z}\}$$

is an irreducible semigroup that has the properties (2) and (3). Since $\mathrm{tr}\,(A) > 0$ and $\mathrm{tr}\,(A^2) = 0$, A is not similar to a nonnegative matrix. Indeed, if A was similar to a nonnegative matrix B, then at least one of diagonal elements of B would be nonzero, as $\mathrm{tr}\,(B) = \mathrm{tr}\,(A) > 0$. But, then we would have $\mathrm{tr}\,(A^2) = \mathrm{tr}\,(B^2) > 0$. This contradiction shows that $\mathcal{S}$ satisfies (4). It is easy to see that $A^{2k} = C^{2k}$ and $A^{2k-1} = 2P - C^{2k-1}$ for all $k \in \mathbb{Z}$. Therefore, the trace of any (positive or negative) power of A is nonnegative if the dimension n is even. Hence, the proof is finished in this case, as $\mathcal{S}$ also satisfies the condition (1). If n is odd, we must define another matrix A.

Assume now that the dimension n is odd, and let $w = e^{2\pi i/n}$ be the primitive nth root of unity. Then the matrix C can be diagonalized as $C = UDU^*$, where D is the diagonal matrix $D = \mathrm{diag}(1, w, w^2, \ldots, w^{n-1})$ and $U = (u_{jk})_{j,k=1}^n$ is the unitary matrix with the entries $u_{jk} = w^{(j-1)(k-1)}$. We will determine a polynomial p of degree at most $n-1$ with real coefficients such that the circulant matrix $A = p(C)$ satisfies the conditions

$$\mathrm{tr}\,(A) > 0, \quad \mathrm{tr}\,(A^2) = 0 \quad \text{and } \mathrm{tr}\,(A^k) \geqslant 0 \quad \text{for all } k \geqslant 3,$$

so that all positive powers of A have nonnegative diagonal entries. By the Lagrange interpolation formula, there exists a unique polynomial p of degree at most $n-1$ such that

$$p(1) = 1, \quad p(w^k) = \frac{i}{\sqrt{n-1}} \quad \text{and}$$
$$p(\overline{w}^k) = -\frac{i}{\sqrt{n-1}} \quad \text{for all } k = 1, 2, 3, \ldots, \frac{n-1}{2}.$$

$\left(\text{If } n = 3, \text{ then } p(z) = \left(\frac{1}{3} - \frac{1}{\sqrt{6}}\right)z^2 + \left(\frac{1}{3} + \frac{1}{\sqrt{6}}\right)z + \frac{1}{3}.\right)$ Since $p(\overline{z}) = \overline{p(z)}$ for all complex numbers z, all the coefficients of p are real numbers. Therefore, the matrix $A = p(C)$ is a real circulant matrix with the spectrum $\left\{1, \frac{i}{\sqrt{n-1}}, -\frac{i}{\sqrt{n-1}}\right\}$, where the multiplicity of each of the last two eigenvalues is $(n-1)/2$. For all $k \in \mathbb{N}$ we have

$$\mathrm{tr}\,(A^{2k-1}) = 1 \ \text{ and } \ \mathrm{tr}\,(A^{2k}) = 1 + \frac{(-1)^k}{(n-1)^{k-1}}.$$

As before, A is not similar to a nonnegative matrix, since $\mathrm{tr}\,(A) > 0$ and $\mathrm{tr}\,(A^2) = 0$. For each $x \in \mathbb{C}^n$ we have $\|Ax\| \leqslant \|x\|$, $\|A^Tx\| \leqslant \|x\|$, $e^T(Ax) = (A^Te)^Tx = e^Tx$, and $e^T(A^Tx) = (Ae)^Tx = e^Tx$, so that both A and A^T leave the cone X invariant. Since A^T is a circulant matrix commuting with A, the set

$$\mathcal{S} = \mathcal{S}_1 \cup \{A^j(A^T)^k : j, k \in \mathbb{N} \cup \{0\}\}$$

is an irreducible semigroup that has the properties (2)–(4). It is easy to show that the trace of $A^j(A^T)^k$ $(j, k \in \mathbb{N})$ equals 1 if $j + k$ is odd, and $1 \pm (n-1)^{1-(j+k)/2}$ if $j + k$ is even, so that it is always nonnegative. Thus, the condition (1) also holds, and the proof is complete. $\square$

3. Operators on $\mathbb{R}^2$

The theorems of the last section suggest that the case of 2×2 matrices may not be entirely trivial, and this is in fact true.

Theorem 3. *If all members of an irreducible semigroup $\mathcal{S}$ in $M_2(\mathbb{R})$ have nonnegative entries in the positions $(1, 1)$, $(1, 2)$ and $(2, 2)$, then the $(2, 1)$ entry is also nonnegative for all members of $\mathcal{S}$.*

Proof. With no loss of generality we can assume that the semigroup $\mathcal{S}$ is closed for the nonnegative linear combinations of its members.

Suppose that the semigroup $\mathcal{S}$ contains a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, d \geqslant 0$ and $c < 0$. We can assume that $b > 0$. Indeed, since the semigroup is irreducible, it contains a matrix B Wih positive $(1, 2)$ entry. Then for a suitable $\lambda > 0$ the linear combination $A' = B + \lambda A$ has positive $(1, 2)$ entry and negative $(2, 1)$ entry. Therefore, if $b = 0$, we replace the matrix A by the matrix A'.

Now, we can also assume that $c = -1$, otherwise we further replace the matrix A by an appropriate nonnegative multiple of A. Then we have

$$A^2 = \begin{pmatrix} a^2 - b & (a + d)b \\ -a - d & d^2 - b \end{pmatrix}.$$

If we had $a + d = 0$, then $a = d = 0$,

$$A^2 = \begin{pmatrix} -b & 0 \\ 0 & -b \end{pmatrix},$$

and so $b = 0$. This shows that $a + d > 0$.

We define a mapping Φ on the subset of $\mathcal{S}$ consisting of all matrices whose $(1, 2)$ entry equals b and $(2, 1)$ entry equals -1 by

$$\Phi(A) = \frac{1}{a + d} A^2 = \begin{pmatrix} a' & b \\ -1 & d' \end{pmatrix}.$$

Now, define a sequence of matrices by setting $A_0 = A$ and $A_{n+1} = \Phi(A_n)$, $n \in \mathbb{N} \cup \{0\}$. If

$$A_n = \begin{pmatrix} a_n & b \\ -1 & d_n \end{pmatrix},$$

then we shall prove inductively that $0 \leqslant a_n \leqslant a - n \cdot \frac{b}{a+d}$ and $0 \leqslant d_n \leqslant d - n \cdot \frac{b}{a+d}$ which obviously leads into a contradiction, and this completes the proof.

For $n = 0$ we have $a_0 = a$, $d_0 = d$ and the inequalities hold.

If the inequalities hold for n, then $0 \leqslant a_n + d_n \leqslant a + d$, so that

$$\begin{aligned} a_{n+1} &= \frac{a_n^2}{a_n + d_n} - \frac{b}{a_n + d_n} \leqslant a_n - \frac{b}{a_n + d_n} \leqslant \\ &\leqslant a - n \cdot \frac{b}{a + d} - \frac{b}{a + d} = a - (n + 1) \cdot \frac{b}{a + d} \end{aligned}$$

and similarly

$$d_{n+1} \leqslant d - (n + 1) \cdot \frac{b}{a + d}. \quad \square$$

Before stating the next result we give a simple related example.

Example 4. There exists an irreducible semigroup $\mathcal{S}$ in $M_2(\mathbb{R})$ such that every member of $\mathcal{S}$ has nonnegative entries in the positions $(1, 1)$, $(1, 2)$ and $(2, 1)$ and there is a matrix $A \in \mathcal{S}$ with negative $(2, 2)$ entry.

Proof. We write $e = (1, 1)^T$ and $e_1 = (1, 0)^T$ and define the cone $X = \mathbb{R}^+e + \mathbb{R}^+e_1$. Let $\mathcal{S}_1$ be the irreducible semigroup of all rank-one nonnegative matrices xy^T with $x, y \in X$. For the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we have $Ae = 2e_1$ and $Ae_1 = e$, and so A preserves the cone X. Now, the semigroup $\mathcal{S}$ generated by the semigroup $\mathcal{S}_1$ and the matrix A has the desired property, since $A^{2n} = 2^nI$ and $A^{2n+1} = 2^nA$. $\square$

Theorem 5. *Let $\mathcal{S}$ be an irreducible semigroup in $M_2(\mathbb{R})$ such that every member has nonnegative entries in the positions $(1, 1)$, $(1, 2)$ and $(2, 1)$. Then $\mathcal{S}$ is similar to a semigroup of nonnegative matrices.*

Proof. Suppose that the semigroup $\mathcal{S}$ contains a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $d < 0$. We can assume that $\mathcal{S}$ is a norm closed subset of $M_2(\mathbb{R})$ and that it is closed for nonnegative linear combinations of matrices. Therefore, we can assume that $d = -1$. By irreducibility of $\mathcal{S}$ we can find a matrix B with a positive entry in the position $(2, 1)$. Since AB has a nonnegative entry in the position $(2, 1)$, we must have that $c > 0$. We define

$$\mathcal{S}_{-1} = \left\{ A \in \mathcal{S} : \ A = \begin{pmatrix} a & b \\ c & -1 \end{pmatrix} \right\}.$$

For matrices $A, A' \in \mathcal{S}_{-1}$

$$A = \begin{pmatrix} a & b \\ c & -1 \end{pmatrix}, \quad A' = \begin{pmatrix} a' & b' \\ c' & -1 \end{pmatrix},$$

we get

$$AA' = \begin{pmatrix} * & * \\ ca' - c' & * \end{pmatrix}.$$

It follows that $c' \leqslant ca'$. Let $c_0 = \inf_{A \in \mathscr{S}_{-1}} c$. If $c_0 = 0$ then $c' = 0$ which is not the case. So, we must have $c_0 > 0$ and $c \leqslant c_0a$ for each member of $\mathcal{S}_{-1}$.

We define the cone $X = \{(x, y)^T \mid x, y \geqslant 0, \ y \leqslant c_0x\}$, which is in fact the cone generated by the vectors $e_1 = (1, 0)^T$ and $e = (1, c_0)^T$. We now prove that in the basis $\mathcal{B} = \{e_1, e\}$ all the matrices from $\mathcal{S}$ convert into nonnegative matrices.

For a matrix

$$A = \begin{pmatrix} a & b \\ c & -1 \end{pmatrix} \in \mathcal{S}_{-1},$$

we get $Ae_1 = (a, c)^T$ and $Ae = (a + bc_0, c - c_0)^T$. Since $c \leqslant c_0a$, we have $(a, c)^T \in X$, so that Ae_1 and Ae lie in the cone X. It follows that in the basis $\mathcal{B}$, the matrix A converts into a nonnegative matrix. Since any matrix from $\mathcal{S}$ which has a negative entry in the position $(2, 2)$ is a positive multiple of a matrix from $\mathcal{S}_{-1}$, in the basis $\mathcal{B}$ all matrices with negative entry in the position $(2, 2)$ convert into nonnegative matrices.

It remains to see what happens with an arbitrary nonnegative matrix

$$B = \begin{pmatrix} x & u \\ y & v \end{pmatrix} \in \mathcal{S},$$

where $x, y, u, v \geqslant 0$. We fix a sequence

$$A_n = \begin{pmatrix} a_n & b_n \\ c_n & -1 \end{pmatrix} \in \mathcal{S}_{-1}$$

such that $\lim_{n\to\infty} c_n = c_0$. Then

$$A_n B = \begin{pmatrix} * & * \\ c_n x - y & c_n u - v \end{pmatrix},$$

so $c_n x - y \geqslant 0$ for all $n \in \mathbb{N}$. It follows that $c_0 x - y \geqslant 0$, and therefore $(x, y)^T$ lies in the cone X. This implies $Be_1 \in X$. To show that Be also belongs to X, we consider two cases:

1. If $c_0 u - v \geqslant 0$, then $(u, v)^T \in X$, and therefore $Be = (x, y)^T + c_0(u, v)^T$ lies in the cone X;
2. If $c_0 u - v < 0$, then there exists an integer $n_0 \in \mathbb{N}$ such that $c_n u - v < 0$ for all $n \geqslant n_0$. Now $\frac{1}{v - c_n u} A_n B \in \mathcal{S}_{-1}$, and so we must have that $\frac{c_n x - y}{v - c_n u} \geqslant c_0$. Taking limit on the left side we get $\frac{c_0 x - y}{v - c_0 u} \geqslant c_0$ or equivalently $c_0(x + c_0 u) \geqslant y + c_0 v$. This again implies that $Be = (x + c_0 u, y + c_0 v)^T$ lies in the cone X.

Since Be_1 and Be both lie in the cone X, in the basis $\mathcal{B}$ the matrix B converts into a nonnegative matrix. This completes the proof. $\square$

4. Operators on $L^2(X, \mu)$

In this section we consider operators on the complex separable Hilbert space $L^2(X, \mu)$, where μ is a σ-finite measure on X. A kernel operator T on $L^2(X, \mu)$ has a **nonnegative diagonal** if its integral kernel $k_T(x, y)$ has the property that $k_T(x, x) \geqslant 0$ for almost every $x \in X$. In the case of a rank-one operator T on $L^2(X, \mu)$ with integral kernel $k_T(x, y) = f(x)g(y)$ (with $f, g \in L^2(X, \mu)$) this condition is equivalent to the condition $\mathrm{tr}\,(TM_{\chi_E}) \geqslant 0$ for every measurable set $E \subseteq X$, where M_ϕ denotes the multiplication operator on $L^2(X, \mu)$ with a function $\phi \in L^\infty(X, \mu)$ and χ_E denotes the characteristic function of E. This follows from the equality

$$\mathrm{tr}\,(TM_{\chi_E}) = \langle \chi_E f, \overline{g} \rangle = \int_E fg \, d\mu = \int_E k_T(x, x) \, d\mu(x).$$

Let $\mathcal{R}_1$ be the set of all kernel operators T on $L^2(X, \mu)$ with integral kernel of the form $k_T(x, y) = f(x)g(y)$ for some nonnegative functions f and g in $L^2(X, \mu)$. It is easy to verify that $\mathcal{R}_1$ is a maximal semigroup of kernel operators on $L^2(X, \mu)$ having nonnegative diagonals and having rank at most one.

Below we record a result which is an infinite-dimensional generalization of the theorem on rank-one matrices ([1, Theorem 3.1]). The more general situation for the constant rank k seems to be more complicated and we could only get partial results. However, we think the problem is worth pursuing further.

Theorem 6. *Let $\mathcal{S}$ be a maximal irreducible semigroup of kernel operators on $L^2(X, \mu)$ having nonnegative diagonals and having rank at most one. Then there exists a unitary multiplication operator M_ϕ on $L^2(X, \mu)$ such that*

$$\mathcal{S} = M_\phi^* \mathcal{R}_1 M_\phi.$$

Proof. By the maximality, the semigroup $\mathcal{S}$ is closed and it is closed under multiplication by positive numbers. We first prove that if A is a fixed nonzero member of $\mathcal{S}$, then the ideal $\mathcal{S}A\mathcal{S}$ equals $\mathcal{S}$. Let T be any nonzero member of $\mathcal{S}$. To show that $T \in \mathcal{S}A\mathcal{S}$, note that there must be a member $B \in \mathcal{S}$ with TBA nonzero. Indeed, if $T\mathcal{S}A = \{0\}$ then, for any nonzero vector x in the range of A, the closed linear span of the set $\{Sx : S \in \mathcal{S}\}$ is a proper invariant subspace, contradicting the irreducibility of $\mathcal{S}$. Similarly, there is a $C \in \mathcal{S}$ with $(TBA)CT$ nonzero. But since $TBACT$ is a rank-one operator, it is just a positive multiple of T. This shows that $T \in \mathcal{S}A\mathcal{S}$.

Next, fix a nonzero $A \in \mathcal{S}$ and let its kernel be $u(x)v(y)$ with $u, v \in L^2(X, \mu)$. Let $\mathcal{F}$ and $\mathcal{G}$ be sets of functions in $L^2(X, \mu)$ defined by

$$\mathcal{F} = \{Su : S \in \mathcal{S}\} \text{ and } \mathcal{G} = \{\overline{(S^*\overline{v})} : S \in \mathcal{S}\},$$

where $\overline{h}$ denotes the complex conjugate of a function h. Thus every member of $\mathcal{S}$ has an integral kernel of the form $f(x)g(y)$ with $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

Because of irreducibility there is no set of positive measure on which all functions in $\mathcal{F}$ are zero. Similarly for $\mathcal{G}$. Now, after fixing representatives for all $f \in \mathcal{F}$, for each $f \in \mathcal{F}$ define ϕ_f by

$$\phi_f(x) = \frac{\overline{f(x)}}{|f(x)|} \text{ if } f(x) \neq 0. \tag{1}$$

Note that this is partially defined and its domain is $D_f = \{x \in X : f(x) \neq 0\}$. We claim that $\phi_f = \phi_{f'}$ on the intersection $D_f \cap D_{f'}$. To see this, call the intersection E, and just observe that for each $g \in \mathcal{G}$ the functions fg and $f'g$ are both positive on E, implying that f and f' have the same signum on the subset of E where g is non-zero. Because of irreducibility we now conclude that f and f' have the same signum almost everywhere on E, and so the claim is proved.

Since the semigroup $\mathcal{S}$ is a separable set in the Banach algebra of all compact operators, there is no loss of generality in assuming that the union D of all D_f with $f \in \mathcal{F}$ is a measurable set. The complement of D has measure zero, so that the function ϕ given by (1) is well defined on X. Then $\mathcal{T} = M_\phi \mathcal{S} M_\phi^*$ is a semigroup of rank-one operators. Now the set $\{\phi f : f \in \mathcal{F}\}$ consists of nonnegative functions whose supports span X. Since members of $\mathcal{T}$ have nonnegative diagonals, we conclude that they have nonnegative integral kernels. By maximality, $\mathcal{T} = \mathcal{R}_1$. This completes the proof. □

References

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